THE INTEGRAL AS THE LIMIT OF A SUM, AND A THEOREM OF DUHAMEL'S

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1. Introduction. A satisfactory treatment of the integral as the limit of a sum is to be found in any standard text-book on the calculus. If $f(x)$ is a real function of the real variable x, continuous throughout the interval $a \leqq x \leqq b$, then the sum

$$f(x_1')\Delta x_1 + f(x_2')\Delta x_2 + \cdots + f(x_n')\Delta x_n,$$

where $\Delta x_i = x_i - x_{i-1}$, $x_0 = a$, $x_n = b$, and $x_{i-1} \leqq x_i' \leqq x_i$, converges toward a limit when n becomes infinite, the longest Δx_i converging at the same time toward zero. Furthermore this limit, the definite integral of $f(x)$ taken between the limits a and b:

$$\int_a^b f(x)\,dx,$$

may be computed in terms of the indefinite integral

$$F(x) = \int f(x)\,dx$$

by means of the theorem that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This and the corresponding theorem for multiple integrals, I say, are treated satisfactorily in standard treatises on the calculus.* The treatment of the application of these theorems to problems in physics and mechanics, both in courses and in text-books on the calculus, is however far from satisfactory, the attitude of the mathematician often being that these applications are without value for mathematics, while for the physicist any reasoning is good enough which in the long run leads to the right formula. To throw rigor to the winds as soon as the hypnotic influence of the environment of a course in

*For an excellent treatment of double and multiple integrals *cf.* Goursat, *Cours d'analyse mathématique*, Paris, 1902, ch. 6.

pure mathematics ceases is to regard rigor as a frill or as a luxury, as something having at most aesthetic reasons for existence,—not as a habit of thought, of practical value in research work. On the other hand the method to which this paper is devoted offers a valuable means of training students to appreciate the meaning of the integral calculus, and moreover a means which is alike valuable to students of applied and to students of pure mathematics. It quickens interest and develops power.

The method appears to be due to Duhamel* and it will be referred to in this paper as Duhamel's Theorem. It is contained in the principle (*cf.* §2 below) that in the limit of a sum of infinitesimals, any infinitesimal may be replaced by one that differs from it by one of higher order. It was introduced into calculus teaching in this country largely through Professor Byerly's, and Professor B. O. Peirce's text-books† and teaching.

Objections have been raised by Mansion to the ordinary formulation of the theorem.‡ His criticism is chiefly destructive, for while he points out how the theorem may be interpreted so as to be correct, the formulation to which he is led is, as he himself observes, of no use in practice, and his conclusion is that for this reason the only value which the theorem retains is for heuristic purposes,—as a "principe d'invention." In fact, what Mansion has done in this paper is similar to what Seidl§ did in pointing out that a convergent series of continuous functions does not necessarily define a continuous function,‖ and in finding out at what point Cauchy's proof of the contrary theorem breaks down. But just as Seidl failed to construct from his observations on series the principle of uniform convergence, and to introduce that principle as a means of investigation,—it was Weirstrass¶ who some seven years earlier, in two papers not published at the time, had taken this step; later, through his university lectures, the method became generally known,—so Mansion fails to extract from Duhamel's principle as originally stated a theorem formulated

* Mansion (*cf.* below) refers it to Duhamel, who gives it in his *Eléments de calcul infinitésimal*, Paris, 1856, p. 35, and applies it to numerous problems in geometry and mechanics.

† Byerly, *Differential Calculus*, Boston, 1882, ch. x, Infinitesimals; *Problems in Differential Calculus*, Boston, 1895, ch. x B. O. Peirce, *Newtonian Potential Function*, 1st ed., Boston, 1886.

‡ *Mathesis*, vol. 8 (1888), p. 149.

§ *Abh. der kgl. bayerischen Akad. der Wiss.* (math.-phys. Cl.), vol. 5 (1848), p. 381.

‖ This fact had already been noted by Abel in his memoir on the binomial series, *Journ. für Math.*, vol. 1 (1826), p. 316, in the case of the Fourier's series $\Sigma(-1)^{n-1}\sin nx/n$; and it had also been observed by Stokes almost simultaneously with Seidl.

¶ *Cf.* two papers of the years 1841 and 1842, *Werke*, vol. 1, p. 67 and p. 75.

so simply that it can easily be applied to the cases that arise in practice, and thus to rescue Duhamel's principle for the important place in the calculus which it was designed to occupy. To put Duhamel's principle on a firm foundation and to make clear its importance in the applications of the calculus is the object of the present paper.

2. Duhamel's Theorem. Duhamel gives the following

THEOREM. Let $a_1 + a_2 + \cdots + a_n$ *be a sum of positive infinitesimals* which approaches a limit when* $n = \infty$. Let $\beta_1 + \beta_2 + \cdots + \beta_n$ *be a second sum of positive infinitesimals which differ respectively from the infinitesimals of the first sum by infinitesimals of higher order; i. e. let*

$$\lim_{n=\infty} \frac{\beta_i}{a_i} = 1.$$

Then the second sum approaches a limit when $n = \infty$, *and this limit is the same as that of the first sum:*

$$\lim_{n=\infty} (\beta_1 + \beta_2 + \cdots + \beta_n) = \lim_{n=\infty} (a_1 + a_2 + \cdots + a_n).$$

To prove the theorem let

$$\frac{\beta_i}{a_i} = 1 + \epsilon_i, \qquad \beta_i = a_i + \epsilon_i a_i,$$

and let the numerical value of the largest ϵ_i be denoted by ϵ:

$$|\epsilon_i| \leqq \epsilon, \qquad i = 1, 2, \cdots n.$$

Then

$$\begin{array}{ccccc} a_1 - \epsilon a_1 & \leqq & \beta_1 & \leqq & a_1 + \epsilon a_1 \\ a_2 - \epsilon a_2 & \leqq & \beta_2 & \leqq & a_2 + \epsilon a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n - \epsilon a_n & \leqq & \beta_n & \leqq & a_n + \epsilon a_n \end{array}$$

Adding, we get

$$(1 - \epsilon)\sum_{i=1}^{n} a_i \leqq \sum_{i=1}^{n} \beta_i \leqq (1 + \epsilon)\sum_{i=1}^{n} a_i.$$

Since $\lim \epsilon = 0$ when $n = \infty$, each extreme of this double inequality approaches as its limit $\lim \Sigma a_n$, and hence $\Sigma \beta_n$ approaches this limit also, *q. e. d.*

* By an infinitesimal is meant a *variable* whose *limit* is zero.

3. Applications. (*a*) *Attraction of a Rod.* Let it be required to find the attraction of a straight rod of variable density on a particle situated in the line of the rod.

Here we start with the physical law of gravitation, namely that two particles of masses m and m' attract each other with a force f proportional to their masses and inversely proportional to the square of the distance r between them:

$$f = \kappa \frac{mm'}{r^2} .$$

Now the rod is not made up of particles, but is a continuous distribution of matter of density ρ. How can we apply the law of gravitation to it? As follows: Divide it up into n parts, and denote the length of the i-th segment $x_i - x_{i-1}$ by Δx_i. Let ρ_i', ρ_i'' denote respectively the minimum and the maximum densities in this segment. Then the mass ΔM_i of so much of the rod obviously lies between $\rho_i' \Delta x_i$ and $\rho_i'' \Delta x_i$:

$$\rho_i' \Delta x_i \leqq \Delta M_i \leqq \rho''_i \Delta x_i,$$

the equality signs holding in the special case of constant density.

Denote the attraction of the rod which we seek to compute by A, the part of the attraction due to the i-th segment by ΔA_i, so that

$$A = \sum_{i=1}^{n} \Delta A_i .$$

Now we can approximate to ΔA_i as follows: If all the mass ΔM_i were concentrated in a point at the nearer end of the i-th segment, we should obtain an attraction $\kappa m \Delta M_i / x_{i-1}^2$ greater than the true attraction ΔA_i, and if we were to replace ΔM_i by the larger mass $\rho_i'' \Delta x_i$, the attraction thus obtained would be still greater; hence

$$\Delta A_i < \kappa \frac{m \rho''_i \Delta x_i}{x_{i-1}^2} .$$

On the other hand, if we were to concentrate the smaller mass $\rho_i' \Delta x_i$ in a point at the further end of the i-th segment, we should obtain an attraction $\kappa m \rho_i' \Delta x_i / x_i^2$ smaller than ΔA_i. Hence

$$\kappa \frac{m \rho_i' \Delta x_i}{x_i^2} < \Delta A_i < \kappa \frac{m \rho''_i \Delta x_i}{x_{i-1}^2} \tag{1}$$

Here, m is at the point O: $x = 0$.

The expressions that form the extremes of this double inequality have the further property that they both differ from ΔA_i by a small percent of ΔA_i when Δx_i is small; in other words, either one differs from ΔA_i by an infinitesimal of a higher order; for, divide through, say, by the left hand expression:

$$1 < \frac{\Delta A_i}{\kappa \dfrac{m\rho_i' \Delta x_i}{x_i^2}} < \frac{\rho_i'' x_i^2}{\rho_i' x_{i-1}^2}.$$

When $n = \infty$ and all the Δx_i's approach 0 as their limit, the right hand member of this double inequality approaches 1, hence the middle ratio does, too, and we see that $\kappa m\rho_i' \Delta x_i / x_i^2$ differs from ΔA_i by an infinitesimal of a higher order.

We are now in a position to apply Duhamel's Theorem and thus to evaluate the attraction A by integration. Since

$$A = \sum_{i=1}^{n} \Delta A_i,$$

we may allow n to become infinite, each ΔA_i approaching 0 as its limit, and we shall still have

$$A = \lim_{n=\infty} \sum_{i=1}^{n} \Delta A_i.$$

But here, in dealing with the limit of the sum $\Sigma \Delta A_i$, we may replace ΔA_i by any more convenient infinitesimal which differs from it by an infinitesimal of higher order. If, now, we can find such an infinitesimal, of the form $f(x_i)\Delta x_i$, then the new limit:

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i,$$

can be evaluated by integration, for the fundamental theorem of §1 will apply to it.

Such an infinitesimal is suggested by the form of the approximations in (1); it is:

$$\kappa \frac{m\rho_i \Delta x_i}{x_i^2},$$

where ρ_i is formed for the point $x = x_i$; the function $f(x)$ being

$$f(x) = \kappa \frac{m\rho \Delta}{x^2}.$$

For, dividing (1) through by it:

$$\frac{\rho_i'}{\rho_i} < \frac{\Delta A_i}{\kappa \frac{m\rho_i \Delta x_i}{x_i^2}} < \frac{\rho_i'' x_i^2}{\rho_i x_{i-1}^2},$$

and the limit of each extreme is 1.

Hence by Duhamel's Theorem and the theorem in §1 we infer that

$$A = \lim_{n=\infty} \sum_{i=1}^{n} \Delta A_i = \lim_{n=\infty} \sum_{i=1}^{n} \kappa \frac{m\rho_i \Delta x_i}{x_i^2} = \int_a^b \kappa \frac{m\rho}{x^2} dx,$$

and the attraction of the rod is formulated as a definite integral. If, in particular, ρ is a constant, then

$$A = \kappa m\rho \int_a^b \frac{dx}{x^2} = \kappa m\rho \left(\frac{1}{a} - \frac{1}{b}\right).$$

The mass M of the rod is here $\rho(b - a)$, and thus

$$A = \kappa \frac{mM}{ab}.$$

The rod attracts, therefore, like a particle of the same mass concentrated, not at the centre of gravity of the rod, but at a distance from O which is a mean proportional detween a and b.

(b) *Attraction of Any Material Body*. Let us now determine the attraction of an arbitrary material body on a particle. We begin by analyzing the method.

Let a system of Cartesian coordinates be assumed, and let the coordinates of the particle m be denoted by (a, b, c). We wish to find the components X, Y, Z along these axes of the attraction which the body exerts on m. Begin with the component X. Divide the volume V occupied by the body into n small pieces $\Delta V_i \ldots \Delta V_n$ and denote the component attraction of the mass ΔM_i contained in the volume ΔV_i by ΔX_i. Then

$$X = \sum_{i=1}^{n} \Delta X_i.$$

The method is now this. If we allow n to become infinite and divide V up in any manner such that the maximum diameter of each ΔV_i approaches 0 as its limit, then, of course,

$$X = \lim_{n=\infty} \sum_{i=1}^{n} \Delta X_i. \tag{2}$$

This limit, as it stands, we do not know how to compute; but we do know how to compute the limit of a sum of the following form:

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta V_i, \tag{3}$$

where $f(x, y, z)$ denotes a function continuous throughout V. For, by the fundamental theorem of the integral calculus* referred to in §1, this limit exists and is the triple integral of $f(x, y, z)$ extended throughout V:

$$\iiint\limits_V f(x, y, z)\, dV.$$

It may be computed by means of three successive simple integrations; for example, by the formula,

$$\int_a^b dx \int_{y_0}^{y_1} dy \int_{z_0}^{z_1} f(x, y, z)\, dz.$$

If, then, we can succeed in throwing the limit (2) by means of Duhamel's Theorem into the form of the limit (3), then we can evaluate (2) by simple integration. This can, in fact, be done, and we turn now to the proof that it can be. Assume for the present that the body lies wholly to the right of the plane $x = 0$; *i. e.* that at no point of the body x is negative.

We shut ΔX_i in between two expressions differing each from ΔX_i by a small percent of ΔX_i. The form of these approximations will suggest the form of the function $f(x, y, z)$ to be employed. Let ρ_i'', r_i'', α_i'' denote respectively the maximum values of the density ρ in ΔV_i, the distance from (a, b, c) to a point of ΔV_i, and the angle α which a ray drawn from (a, b, c) to a point of ΔV_i makes with the positive axis of x; and let ρ_i', r_i', α_i' denote respectively the minimum values of these functions. Then it is clear that if a particle of mass ΔM_i were concentrated at the shortest distance r_i' from m to a point of ΔV_i, and were

* *Cf.*, for example, Goursat, ch. 7.

placed, not necessarily at a point of ΔV_i, but on a ray making the minimum angle a'_i with the x-axis, the component attraction

$$\kappa \frac{m \Delta M_i}{r'^2_i} \cos a'_i$$

would be greater than ΔX_i; and if ΔM_i were replaced by the larger value $\rho''_i \Delta V_i$ — we have as before

$$\rho'_i \Delta V_i \leqq \Delta M_i \leqq \rho''_i \Delta V_i,—$$

then the inequality would only be strengthened. In a similar manner we obtain an expression that is less than ΔX_i, and hence the double inequality

$$\kappa \frac{m \rho'_i \Delta V_i}{r''^2_i} \cos a''_i < \Delta X_i < \kappa \frac{m \rho''_i \Delta V_i}{r'^2_i} \cos a'_i. \tag{4}$$

The first and last members above suggest a form of the summand in (3) such as is desired, namely, the summand obtained by setting

$$f(x, y, z) = \kappa \frac{m\rho}{r^2} \cos a.$$

For, dividing (4) through by this function formed for any arbitrary point of ΔV_i and multiplied by ΔV_i, we get

$$\frac{\rho'_i r_i^2 \cos a''_i}{\rho_i r''^2_i \cos a_i} < \frac{\Delta X_i}{\kappa \frac{m \rho_i}{r_i^2} \cos a_i} < \frac{\rho''_i r_i^2 \cos a'_i}{\rho_i r'^2_i \cos a_i}. \tag{5}$$

When ΔV_i shrinks down toward a point as its limit, each extreme of this double inequality approaches 1 as its limit, and hence the middle term must do so also.

Thus the conditions of Duhamel's Theorem, §2, are satisfied and we have

$$\begin{aligned} X &= \iiint\limits_V \kappa \frac{m\rho \cos a}{r^2}\, dV \\ &= \kappa m \iiint\limits_V \frac{\rho (x-a)\, dx\, dy\, dz}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^{3/2}}, \end{aligned} \tag{6}$$

with similar formulas for Y and Z.

If the solid lies wholly to the left of the plane $x = 0$, the component attraction in the direction of the negative axis of x can be computed in precisely the same manner; and if the solid lies on both sides of the plane, then the

attraction can be computed for each piece separately and the results, affected by the proper algebraic signs, added. The final formula will be the same as the above, (6).

4. Critique of Duhamel's Theorem and its Application in §3. The infinitesimals α_i, β_i that appear in Duhamel's Theorem depend on two variables, i and n, and the *hypothesis*

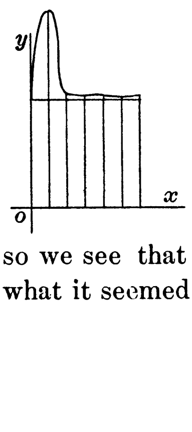
$$\lim_{n=\infty} \frac{\beta_i}{\alpha_i} = 1$$

tacitly assumes that i varies with n according to some specified law, or perhaps according to any law whatever; it is not explicit on this point. The *proof* tacitly assumes that $\lim \epsilon = 0$ when $n = \infty$.

An example will make clear questions which a critical examination of the law and its proof suggest.* Consider the curve

$$y = 1 + n^2 x e^{-n^2x^2} \tag{7}$$

in the interval $0 \leqq x \leqq 1$. The graph of the curve is as indicated in the accompanying figure. For any specified value of x in the interval the ordinate



of the curve approaches 1 as its limit when $n = \infty$; but the peak rises higher with increasing values of n and moves at the same time further and further to the left toward the axis of ordinates. At first sight it might appear as if the area under the curve necessarily approached as its limit the area under the limiting curve, $y = 1$; *i. e.* the area 1. But another factor comes in, namely, the ever increasing height of the peak, and so we see that it is not certain that the limit of the area under the curve is what it seemed to be. In fact, let us determine it; it is:

$$A_n = \int_0^1 y\,dx = \int_0^1 (1 + n^2 x e^{-n^2x^2})\,dx$$

$$= (x - \tfrac{1}{2} e^{-n^2x^2})\Big|_0^1 = 1 + \tfrac{1}{2}(1 - e^{-n^2}),$$

and $\lim A_n$ is seen to be $1\frac{1}{2}$ instead of 1.

Let us now divide the interval (0, 1) into n equal parts and through the points of division draw lines parallel to the axis of ordinates, thus dividing the

* This is the geometric form of an example given by Mansion, *l. c.*

area under the curve (7) into strips. As n increases, some of these strips will run up into the peak of the curve, so that their maximum altitude will increase indefinitely with n. In fact, let $\Delta x_i = 1/n$, $x_i = i\Delta x_i$. Then

$$y_i = 1 + nie^{-i^2}$$

and if i is held fast while n increases, y_i becomes infinite with n.

As the infinitesimal α_i of Duhamel's Theorem we will take the area of the i-th one of the above strips, namely,

$$\begin{aligned} \alpha_i &= \int_{x_{i-1}}^{x_i} (1 + n^2 x e^{-n^2x^2})\,dx \\ &= \left(x - \tfrac{1}{2}\, e^{-n^2x^2}\right)\Big|_{x_{i-1}}^{x_i} \\ &= \Delta x_i + \tfrac{1}{2}\,(e^{-(i-1)^2} - e^{-i^2})\,; \end{aligned}$$

as β_i we will take the part of the area of this strip lying below the limiting curve $y = 1$, namely,

$$\beta_i = \Delta x_i.$$

Then if we fix our attention on the strip that contains a given point, as for example, the point $x = \lambda$, $y = 0$, and allow n to become infinite, i will become infinite at the same time and x_i, x_{i-1} will both approach λ as their limit. Hence we shall have

$$\lim_{n=\infty} \frac{\beta_i}{\alpha_i} = \lim_{n=\infty} \frac{1}{1 + \frac{n}{2}\left(e^{-n^2x_{i-1}^2} - e^{-n^2x_i^2}\right)} = 1$$

and the conditions of Duhamel's Theorem, as they might readily be understood, are fulfilled. But the theorem would then be false, for

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \Delta x_i + \sum_{i=1}^{n} \tfrac{1}{2}[(1 - e^{-1}) + (e^{-1} - e^{-4}) + \cdots + (e^{-(n-1)^2} - e^{-n^2})]$$

and
$$\lim_{n=\infty} \sum_{i=1}^{n} \alpha_i = 1\tfrac{1}{2}.$$

The explanation is simple. For the case in hand the maximum ϵ_i, taken numerically, namely ϵ, does not approach zero. Or again, it is possible to allow i so to vary with n that β_i/α_i does not approach 1 ; for if we hold i fast, and express Δx_i as $\dfrac{1}{n}$, then $\dfrac{\beta_i}{\alpha_i} = [1 + \dfrac{n}{2}\,(e^{-(i-1)^2} - e^{-i^2})]^{-1}$ and β_i/α_i approaches

0 as its limit when n becomes infinite. Here, the number of the strip remains constant and the strip moves further and further to the left.*

It is clear from this example that the formulation and the proof of the theorem leave something to be desired.†

The foregoing critique applies to Duhamel's Theorem itself. Let us now turn to its application in §3. First, the restriction that all the infinitesimals must be of like sign made a clumsy division into special cases necessary, the general result being identical with that obtained in each of the special cases.

A more serious difficulty is the following. Suppose the body extends up to the plane $x=0$. Then for a ΔV_i abutting on this plane $a_i'' = \frac{1}{2}\pi$ and $\cos a_i'' = 0$. Hence it is not true in this case that the limit of the left hand member of (5) is unity, and the hypotheses of Duhamel's Theorem are not fulfilled. It would be possible, to be sure, to take the boundary of the solid distinct from the plane $x=0$, but near to it, and to allow the boundary subsequently to approach this plane as a limit. But to be obliged to take such precautions as these and go through with the consideration of a passage to a limit would detract very largely from the simplicity which is desirable in the case of so important a theorem. For the reasons here given we modify the statement of Duhamel's theorem in such a manner that it will apply to the cases that arise in practice and at the same time retain a satisfactory degree of simplicity.

5. Revised Formulation of Duhamel's Theorem. The important applications of Duhamel's Theorem in practice are to cases in which we wish to show that a given limit of a sum:

$$\lim_{n=\infty} \sum_{i=1}^{n} a_i$$

* This example is closely related to the question of uniform convergence and the integration of a series term by term; *cf.* the author's geometric treatment of this subject, *Bulletin Amer. Math. Soc.*, ser. 2, vol. 3 (1896), p. 59.

† The formulation may be made precise and the theorem correct by requiring that $\lim \beta_i/a_i = 1$, *no matter how i varies with n*, as n becomes infinite. For it can then be readily shown that $\lim_{n=\infty} \epsilon = 0$, and thus the proof of §2 will hold. But this formulation is of little use for practical purposes, since there is no simple sufficient condition which insures that the hypotheses of the theorem are fulfilled; and even if such a test were found, the theorem would still be inapplicable to simple cases that arise in practice, as will presently appear. The formulation given below (§5), on the other hand, is not open to this objection.

is equal to a definite integral,* either to

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)\,dx,$$

or to

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta S_i = \iint\limits_S f(x,y)\,dS,$$

or to

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i = \iiint\limits_V f(x, y, z)\,dV.$$

We will restrict ourselves for simplicity to the first case.

Definition. Let ζ_i be defined by the equation

$$a_i = f(x_i)\Delta x_i + \zeta_i \Delta x_i.$$

Then the infinitesimal a_i is said to differ uniformly from the infinitesimal $f(x_i)\Delta x_i$ by an infinitesimal of higher order than Δx_i if, a positive quantity ϵ being chosen arbitrarily small, it is then possible to find a second positive constant δ such that, no matter how the interval $a \leqq x \leqq b$ be divided into n parts, we always have

$$|\zeta_i| < \epsilon,$$

provided merely that the division has proceeded far enough so that each $\Delta x_i < \delta$. Here a_i may be positive, negative, or zero.† In the case of a double or triple integral ζ_i will be defined by the equation

$$a_i = f(x_i, y_i)\Delta S_i + \zeta_i \Delta S_i \qquad \text{or} \qquad a_i = f(x_i, y_i, z_i)\Delta V_i + \zeta_i \Delta V_i,$$

where ΔS_i, ΔV_i denote respectively a piece of the surface or the volume into which the region of integration has been divided.

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*The notation here is not the same as in §2, the present a_i corresponding to the former β_i and the present $f(x_i)\Delta x_i$ to the former a_i.

†If one wishes to admit negative Δx_i's to the consideration, then the last relation must read

$$|\Delta x_i| < \delta.$$

THEOREM. *Let*

$$a_1 + a_2 + \cdots + a_n \tag{A}$$

be a sum of infinitesimals and let a_i *differ uniformly by an infinitesimal of higher order than* Δx_i *from the summand* $f(x_i)\Delta x_i$ *of the definite integral*

$$\int_a^b f(x)\,dx \tag{B}$$

of the function $f(x)$, *this function being continuous throughout the interval* $a \leqq x \leqq b$. *Then the sum* (A) *approaches a limit when* $n = \infty$, *and the value of this limit is the definite integral* (B) *:*

$$\lim_{n=\infty} \sum_{i=1}^{n} a_i = \int_a^b f(x)\,dx.$$

Proof. Since

$$a_i = f(x_i)\Delta x_i + \zeta_i \Delta x_i$$

and

$$|\zeta_i| < \epsilon, \qquad \Delta x_i < \delta,$$

it follows that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} f(x_i)\Delta x_i + \sum_{i=1}^{n} \zeta_i \Delta x_i,$$

$$\left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} f(x_i)\Delta x_i\right| \leqq \sum_{i=1}^{n} |\zeta_i| \Delta x_i < \sum_{i=1}^{n} \epsilon \Delta x_i = \epsilon(b-a), \tag{8}$$

when $\Delta x_i < \delta$. Now the sum $\Sigma f(x_i)\Delta x_i$ converges toward the integral (B) when $n = \infty$, and so

$$\left|\sum_{i=1}^{n} f(x_i)\Delta x_i - \int_a^b f(x)\,dx\right| < \epsilon, \tag{9}$$

when $\Delta x_i < \delta'$. If δ is chosen at least as small as δ', relations (8) and (9) will hold simultaneously as soon as $\Delta x_i < \delta$, $i = 1, \cdots n$. From (8) and (9) we infer that

$$\left|\sum_{i=1}^{n} a_i - \int_a^b f(x)\,dx\right| < \epsilon(b-a) + \epsilon,$$

when $\Delta x_i < \delta$, and this relation is coextensive with the assertion of the theorem we wished to prove.

Both theorem and proof apply without modification to multiple integrals.

6. Application of Duhamel's Revised Theorem to the Example of §3. The formulation of the problem down to the point of the double inequality (4) is the same as in §3. We now identify the infinitesimal ΔX_i with the a_i of §5 and set

$$f(x, y, z) = \kappa \frac{m\rho}{r^2} \cos \alpha.$$

The functions ρ, r, $\cos \alpha$ are continuous throughout the whole volume V inclusive of the boundary, and r never vanishes in this region. Hence ρ and $r^{-2} \cos \alpha$ are both uniformly continuous throughout V, and the oscillation of each in ΔV_i can be made uniformly small by choosing all the ΔV_i's sufficiently small. If, then, η denote an arbitrarily small positive quantity, the oscillation in each ΔV_i will be less numerically than η for all divisions of V for which the maximum diameter of any ΔV_i is less than a suitably chosen positive constant δ. Moreover the maximum numerical value G of either function in V is finite:

$$|\rho| \leqq G, \qquad \left|\frac{\cos \alpha}{r^2}\right| \leqq G.$$

Thus we shall have

$$\rho_i'' = \rho_i + \theta_i^{(1)},$$

$$\frac{\cos \alpha_i'}{r_i'^2} = \frac{\cos \alpha_i}{r_i^2} + \theta_i^{(2)},$$

where ρ_i, r_i, $\cos \alpha_i$ apply to an arbitrarily chosen point of ΔV_i and

$$|\theta_i^{(1)}| < \eta, \qquad |\theta_i^{(2)}| < \eta.$$

Hence

$$\kappa \frac{m\rho_i'' \Delta V_i}{r_i'^2} \cos \alpha_i' = \kappa \frac{m\rho_i \Delta V_i}{r_i^2} \cos \alpha_i + \zeta_i'' \Delta V_i,$$

where

$$\zeta_i'' = \kappa m (r_i^{-2} \cos \alpha_i \theta_i^{(1)} + \rho_i \theta_i^{(2)} + \theta_i^{(1)} \theta_i^{(2)}) \text{ and } |\zeta_i''| < \kappa m (2G + \eta)\eta = \epsilon.$$

A similar relation holds for the left hand member of (4). Hence if we subtract the expression

$$\kappa \frac{m\rho_i \Delta V_i}{r_i^2} \cos \alpha_i$$

from each member of (4), the new extremes will each be less numerically than $\epsilon \Delta V_i$, and we get

$$\left|\Delta X_i - \kappa \frac{m\rho_i \Delta V_i}{r_i^2} \cos \alpha_i\right| < \epsilon \Delta V_i.$$

This relation expresses precisely the fact that

$$\kappa \frac{m\rho_i \Delta V_i}{r_i^2} \cos a_i$$

differs uniformly from ΔX_i by an infinitesimal of higher order.

7. The Axioms of Physics. *Gravitation.* We have chosen for purposes of illustration a problem in gravitation. The method set forth applies in the same manner to problems in centres of gravity, moments of inertia, fluid pressures, kinetic energy of a rigid body (or, generally, of any continuous distribution of matter), work done on a rigid body by gravitational forces, etc. It is interesting to inquire in each case what the physical laws assumed are,—what is definition and what is proof.

Consider the problem of §3. Here, to begin with, the law of gravitation for two particles is assumed. But is that all? Have we, with the aid of this law alone, *computed* the attraction of the given body? By no means. We have assumed as further axioms

(a) that a resultant attraction for the whole body and for any piece of it, exists;

(b) that when the body is divided up in any way the resultant attraction due to the attractions of all the component pieces is equal to the resultant attraction of the whole body;

(c) that the intensity of the resultant attraction for a given piece is less than the force which a particle of the same mass as this piece would exert if placed at the minimum distance from the given particle to a point of the piece, and greater than if that particle were placed at the maximum distance;

(d) that the direction of the resultant attraction of any piece lies within any convex cone that can be drawn with the given particle as vertex, containing the piece in question.*

Moment of Inertia. An analogous set of axioms is employed in finding centres of gravity of continuous distributions of matter, and moments of inertia may be treated in the same way. Another point of view regarding moments of inertia is as follows. Consider, for example, the moment of inertia of a plane lamina of variable density about an axis perpendicular to its plane and

*This axiom regarding the direction of the resulting force is not the same in form as the one employed in §3; but it is evident that the inequality (4) of that paragraph can be deduced from it.

piercing the plane in the point O. Here we begin with the definition of the moment of inertia of a particle of mass m situated in the plane, about O as

$$I = mr^2,$$

where r denotes the distance of the particle from O, and we define the moment of inertia of a system of such particles as the sum of the moments of inertia of the individual particles. Now instead of considering the moment of inertia of the lamina about O as a fundamental concept and proceeding to compute its value as above suggested, we may regard this moment of inertia as a quantity to be defined and for the purpose of the definition lay down requirements suggested by the physical meaning of the moment of inertia of a particle or system of particles. These requirements we may state as follows: The moment of inertia of the lamina, and of any piece of it, shall be so defined,

(a) that when the lamina is divided up in any way, the sum of the moments of inertia of the component pieces shall equal the moment of inertia of the whole lamina;

(b) that the moment of inertia of a piece shall lie between the moments of inertia of a particle of the same mass when concentrated at the nearest point of the piece and at the farthest point respectively.

The question that now presents itself is whether such a definition is possible, and if so, how the moment of inertia may be computed. Let us see. Suppose that the definition is possible. Denote the moment of inertia of the whole lamina by I, that of one of the component parts into which it is divided by ΔI_i. Then, if ΔM_i, ΔS_i, ρ_i', ρ_i'', r_i', r_i'' denote respectively the mass, the area, the minimum and the maximum densities, and the minimum and the maximum distances from O to a point of ΔS_i, we shall have

$$I = \sum_{i=1}^{n} \Delta I_i = \lim_{n=\infty} \sum_{i=1}^{n} \Delta I_i,$$

$$\rho_i' \Delta S_i \leqq \Delta M_i \leqq \rho_i'' \Delta S_i,$$

$$\rho_i' \Delta S_i r_i'^2 < \Delta I_i < \rho_i'' \Delta S_i r_i''^2.$$

The extremes of the last inequality suggest the function

$$f(x, y) = \rho r^2$$

formed for an arbitrary point (x_i, y_i) of the region ΔS_i. Now ρ and r are both continuous functions of x, y throughout the whole region of the lamina; hence

they are uniformly continuous, and thus we are able to show, just as in §6, that ΔI_i differs uniformly from

$$\rho_i r_i^2 \Delta S_i$$

by an infinitesimal of higher order. All of the conditions of the revised form of Duhamel's Theorem, §5, are fulfilled regarding the infinitesimals $\Delta_i I$ and we see that I must have the value

$$I = \lim_{n=\infty} \sum_{i=1}^{n} \rho_i r_i^2 \Delta S_i = \iint_S \rho r^2 dS.$$

This is a necessary condition for the existence of a moment of inertia which shall satisfy the given requirements. Conversely, it is sufficient, as is readily shown by the law of the mean. Hence we may define the moment of inertia of the lamina as

$$I = \iint_S \rho r^2 dS,$$

and this shall be its definition.

Length of Arc of a Curve. For the elementary student the length of the arc of a curve is a fundamental concept. He is ready to accept the axiom (*i.e.* physical law) that the length of an arc of a plane convex curve, when the arc lies wholly on one side of the straight line which passes through its extremities, is less than the length of a broken line enveloping it and having the same extremities; while on the other hand, the length of the arc is greater than that of the chord. The ordinary formula for the length of an arc of a plane curve may then be proved as in §6. On the other hand, for the professional mathematician the length of the arc of a curve is a matter of definition, and the requirements on which the definition depends lead to considerations analogous to those above discussed in the case of the definition of the moment of inertia.

8. Conclusion. In closing, one word about the place of these theorems in elementary instruction in the calculus. It is advisable that the integral as the limit of a sum be introduced at an early stage and illustrated by numerous problems in the determination of volumes of geometric solids. The solid in question is sliced up and the actual slice is compared with an auxiliary slice whose volume is readily computed in the form

$$f(x_i) \Delta x_i$$

and which is so chosen that the total volume of these auxiliary slices,

$$\sum_{i=1}^{n} f(x_i)\Delta x_i, \tag{10}$$

obviously converges toward the volume sought as a limit. But the limit of (10) can be determined by integration, §1. In many cases, as for example in finding the volume of a regular pyramid or of an anchor ring, a formal proof may readily be given, and it is desirable that some such proofs be given. But the observational faculties of the student should also be exercised and formal proofs, when clumsy, may well be waived in favor of greater stress on the intuitional side of the problem.

Proceeding now to the next stage, the treatment of physical problems, we find that intuition is less clear. The student may be willing to grant the formulation of the attraction of the rod in §2 as the definite integral there found for it, but his reasoning is more likely to be based on formal analogy, such as led him a year earlier to think that $\sin 2x$ is equal to $2 \sin x$, than on an estimate of the relative magnitudes of the infinitesimals with which he is dealing. It is here that Duhamel's Theorem in its original form (§2) gives him a rigorous principle by means of which to determine whether his processes are correct. I say a *rigorous* principle, for the principle meets the highest demands for rigor of which the student is capable at this stage. What is the state of affairs at present regarding the treatment of these physical problems? The infinitesimal is treated as an inconceivably small constant, and an infinitesimal which does not succeed in making itself *persona grata* is neglected. Shall we hesitate to replace this barbarous method by one right in principle and easy of application simply because the whole truth can not be told at this point? The systematic treatment of physical problems by the method of limits and Duhamel's Principle means a step in advance in calculus teaching, and it is possible and desirable for this generation of mathematicians to take this step, both in college courses and in the courses of our technical schools.

Finally the student of mathematical physics and of higher analysis, who has reached the stage at which questions of uniform convergence are treated, will find in the revised form of Duhamel's Theorem a method rigorous according to the highest standards of today.

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